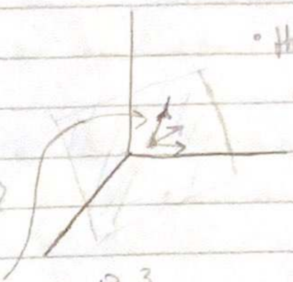
Recap: Dot product

- $\vec{u}$ and $\vec{v}$ are orthogonal iff $\vec{u}\cdot\vec{v} = 0$

12.4: Cross Product



- the plane is a geometric object

Goal: given two vectors

$\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\mathbb{R}^3$

construct a vector $\vec{w} = \langle w_1, w_2, w_3 \rangle \leftarrow \mathbb{R}^3$
that is orthogonal to both $\vec{u}$ & $\vec{v}$

- (we will find $\vec{w}$ canonically)

How? we know

$$\begin{cases} ① \; 0 = \vec{u}\cdot\vec{w} = u_1w_1 + u_2w_2 + u_3w_3 \\ ② \; 0 = \vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3 \end{cases}$$

↳ whatever vector we find will satisfy this condition
(we want to compute $\langle w_1, w_2, w_3 \rangle = \vec{w}$)

>could use elimination ↓

- mult. ① by v_3 & ② by u_3 to get:

$$\begin{cases} 0 = v_3(\vec{u}\cdot\vec{w}) = (u_1v_3)w_1 + (u_2v_3)w_2 + (u_3v_3)w_3 \\ 0 = u_3(\vec{v}\cdot\vec{w}) = (u_3v_1)w_1 + (u_3v_2)w_2 + (u_3v_3)w_3 \end{cases}$$ > same

- subtract ②′ from ①′ ↓

③ $0 = v_3(\vec{u}\cdot\vec{w}) - u_3(\vec{v}\cdot\vec{w})$

$0 = (u_1v_3 - u_3v_1)w_1 + (u_2v_3 - u_3v_2)w_2$

$= -(-(u_1v_3 - u_3v_1)w_1 + (u_2v_3 - u_3v_2)w_2$

Aside

$-ax + by = 0$
let: $x = b$
$y = a$
⇓
$-ab + ba = 0$

shows why we added

∴ ③ has at least the solution

$$\begin{cases} w_1 = u_2v_3 - u_3v_2 \\ w_2 = -(u_1v_3 - u_3v_1) \end{cases}$$

$0 = u_1w_1 + u_2w_2 + u_3w_3$

$= u_1(u_2v_3 - u_3v_2) + u_2(-(u_1v_3 - u_3v_1)) + u_3w_3$

$= u_1\cancel{u_2}v_3 - u_1u_3v_2 - u_1\cancel{u_2}v_3 + u_2u_3v_1 + u_3w_3$

- inputting these to ① we obtain:

⇓

$= u_3(u_2v_1 - u_1v_2 + w_3)$

* either $u_3 = 0$ or $w_3 = u_1 v_2 - u_2 v_1$

Claim: modulo the detail that u_3 may be 0, we have a solution: $\vec{w} = \langle u_2 v_3 - u_3 v_2, (u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$

↳ check it by plugging in (check symbolically)

Determinant: The determinant of the 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- the determinant of the 3 × 3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ is $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Ex compute

$\det \begin{bmatrix} -1 & 3 & 7 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$: solution

$$\det \begin{bmatrix} -1 & 3 & 7 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

* → take product of diagonals & (-) *

$$= -1 \det \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} - 3 \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + 7 \det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= -1(-1 \cdot 1 - 1 \cdot 0) - 3(0 \cdot 1 - 1 \cdot 1) + 7(0 \cdot 0 - (-1)(1))$$

$$= -1(-1) - 3(-1) + 7(1)$$

$$= 1 + 3 + 7 = 11$$

Defn: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$

the cross product of $\vec{u}$ with $\vec{v}$ is:

$$\vec{u} \cdot \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Check: $\vec{u} \cdot \vec{v} = \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

$= (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$

* same as the $\vec{w}$ found on previous page

$\vec{w} = \langle u_2 v_3 - u_3 v_2, (u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$

CROSS PRODUCT

* this has to be done in $\mathbb{R}^3$, much more would be needed to do in $\mathbb{R}^4$ (including more vectors)

- the cross product is a vector operation

(vector in $\mathbb{R}^3$ X vector in $\mathbb{R}^3$ $\Longrightarrow$ vector in $\mathbb{R}^3$)
(cross)

some examples

$\vec{0} \cdot 1$, undefined, 1 is not a vector

$\langle 1, 1 \rangle \times \langle 3, 2 \rangle$, undefined, not in $\mathbb{R}^3$

Proposition (Algebraic properties of cross product):

- Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

scewed communitive

① $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ (showing it the opp. way, but same idea)

proof: $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \vec{i} \begin{vmatrix} v_2 & v_3 \\ u_2 & u_3 \end{vmatrix} - \vec{j} \begin{vmatrix} v_1 & v_3 \\ u_1 & u_3 \end{vmatrix} + \vec{k} \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix}$

$= (v_2 u_3 - v_3 u_2)\vec{i} - (v_1 u_3 - v_3 u_1)\vec{j} + (v_1 u_2 - v_2 u_1)\vec{k}$

$= \langle v_2 u_3 - v_3 u_2, -(v_1 u_3 - v_3 u_1), v_1 u_2 - v_2 u_1 \rangle$

$= -\langle v_3 u_2 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$

$= -\vec{u} \times \vec{v}$

② $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$

③ $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$: distribution on left

④ $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$: distribution on right

⑤ $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ * not associative bcs '•' is a diff. operation than 'x'

⑥ $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ * NOT Associative ∴ ( ) can't shift

Properties (Geometric of cross product):
- Let $\vec{u}, \vec{v} \in \mathbb{R}^3$

① $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ & $\vec{v}$

② $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ w/ θ the angle between $\vec{u}$ & $\vec{v}$

③ $\vec{u} \times \vec{v} = \vec{0}$ iff $\vec{u} \parallel \vec{v}$ (parallel)